MATH3210 - SPRING 2024 - SECTION 004

HOMEWORK 3 - *SOLUTIONS*

Problem 1 (20 points)**.** Let $A, B \subset \mathbb{R}$ be nonempty subsets, and assume that if $x \in A$ or $x \in B$ then $x > 0$. Show that if $A/B = \{x/y : x \in A \text{ and } y \in B\}$, then:

$$\sup(A/B) = \frac{\sup A}{\inf B}$$

whenever $\inf B > 0$.

Solution. Denote $z = \frac{\sup A}{\inf B}$. We will show that z is an upper bound of A/B, and that if $y < z$, then y is *not* an upper bound of A/B.

First, we show that z is an upper bound. If $x \in A/B$, then there exists $a \in A$ and $b \in B$ such that $x = a/b$. Since $\sup A$ is an upper bound of A, $a \leq \sup A$. Similarly, $b \geq \inf B$, and hence $1/b \leq 1/\inf B$ (since all elements of B are positive). Multiplying these inequalities, we see that

$$\frac{a}{b} \leq \frac{\sup A}{\inf B} = z.$$

Hence z is an upper bound.

Now suppose that $y < z$. If $y \leq 0$, y cannot be an upper bound since A and B consist of positive numbers. So without loss of generality, $y > 0$. Let $\varepsilon = \dfrac{\sup A - y \inf B}{1 + y}$. Then $\varepsilon > 0$ since $y < z$. Since $\sup A$ is the least upper bound of A, there exists $a \in A$ such that $a > \sup A - \varepsilon$. Similarly, there exists b such that $b < \inf B + \varepsilon$. Hence

$$\begin{aligned} a/b &> \frac{\sup A - \varepsilon}{\inf B + \varepsilon} \\ &= \frac{(1+y)\sup A - (\sup A - y \inf B)}{(1+y)\inf B + (\sup A - y \inf B)} \\ &= \frac{y(\sup A + \inf B)}{\inf B + \sup A} \\ &= y. \end{aligned}$$

Thus, y cannot be an upper bound of A/B and $z = \sup(A/B)$. ☐

Problem 2 (30 points, 10 each)**.** Determine whether the sequence converges. If it converges, find its limit and prove that the sequence converges to that limit. If it diverges, prove that it diverges.

(a) $\left\{\dfrac{2n+3}{8n+7}\right\}$

(b) $\left\{\dfrac{n^2 - 100n}{7n+3}\right\}$

(c) $\{\sin(\pi \cdot n)\}$

Solutions.

(a) We claim this sequence converges to $1/4$. This follows from the limit arithmetic theorem (aka, the main limit theorem):

$$\lim_{n\to\infty} \frac{2n+3}{8n+7} = \lim_{n\to\infty} \frac{2 + 3/n}{8 + 7/n} = \frac{2 + 3\lim_{n\to\infty} 1/n}{8 + 7\lim_{n\to\infty} 1/n} = 2/8 = 1/4.$$

(b) We claim that the sequence diverges to ∞. Indeed, fix $M > 0$, and let $N = 7M$. Then if $n \geq N$,

$$\frac{n^2 - 100n}{7n+3} \geq \frac{n^2}{7n} \geq \frac{(7M)^2}{7 \cdot (7M)} = M.$$

Hecne the sequence diverges to ∞.

(c) Since $\sin(\pi \cdot n) \equiv 0$, the sequence is constant, and hence converges (to 0).

□

Problem 3 (Book 2.2.11, 20 points)**.** Let (a_n) and (b_n) be sequences, and assume that $b_n \to 0$ and $|a_n| \leq b_n$ for every $n \in \mathbb{N}$. Prove that $a_n \to 0$.

Solution. Let $\varepsilon > 0$. Since $b_n \to 0$, there exists N such that if $n \geq N$, $|b_n - 0| = |b_n| < \varepsilon$. Then if $n \geq N$,

$$|a_n - 0| = |a_n| \leq b_n \leq |b_n| < \varepsilon$$

Hence $a_n \to 0$. □

Problem 4 (10 points)**.** Show that if $I = [a, b]$ is a nonempty interval, and $x, y \in I$, then $|x - y| \leq b - a$.

Solution. If $x, y \in I$, then we have that $a \leq x, y \leq b$. Thus, we also know that $-x, -y \leq -a$. Adding these inquealities on the right in the nontrivial ways, we get that both $x - y \leq b - a$ and $y - x \leq b - a$. Hence $|x - y| \leq b - a$. □

Problem 5 (20 points)**.** Show that if $a_n \to L$ and $|b_n - a_n| \to 0$, then $b_n \to L$.

Solution. Let $\varepsilon > 0$. Since $a_n \to L$, there exists N_1 such that if $n \geq N_1$, $|a_n - L| < \varepsilon/2$. Similarly, there exists N_2 such that if $n \geq N_2$, $|b_n - a_n| < \varepsilon/2$. Set $N = \max\{N_1, N_2\}$. Then if $n \geq N$,

$$|b_n - L| = |(b_n - a_n) + (a_n - L)| \leq |b_n - a_n| + |a_n - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence $b_n \to L$. □